SYMMETRIC ABOUT POLAR AXIS

Theorem-I Prove that a curve given by polar equation is symmetric ~~symmetry~~ with respect to polar axis, if one of the following condition hold.

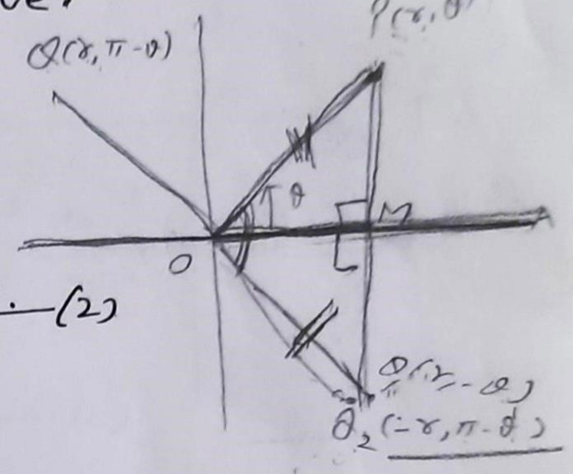
(i) Equation remains unchanged on replacing θ by $-\theta$.

(ii) Equation remains unchanged on replacing r by $-r$ and θ by $\pi-\theta$

OR

State when a polar curve is symmetric with respect to polar axis?. Also prove it

Proof: Let $f(r,\theta)=0$ be the polar curve. (1)



(i) If equation remains unchanged by changing θ by $-\theta$, then we have

$f(r,-\theta)=0$

$\Rightarrow Q_1(r,-\theta)$ also lies on curve ~~(1)~~. —(2)

PQ_1 cuts the polar axis OA at M

ΔPOQ_1 is isosceles triangle with

$OP=OQ_1$

and OM bisects $\angle POQ_1$

$\Rightarrow$ OM is $\perp^r$ bisector of PQ_1

By def. P and Q_1 are in symmetry with respect to polar axis. — (3)

From (2) and (3) we say that curve is symmetric about polar axis.

(ii) If equation (1) remains unchanged on replacing ~~r by -r an~~ r by $-r$ and θ by $\pi-\theta$.

then $f(-r,\pi-\theta)=0$.

$\Rightarrow Q_2(-r,\pi-\theta)$ also lies on the curve. —— (4)

By fig Q_1 and Q_2 are same points.

By case (i), we say that P and Q_2 are in symmetry with respect to polar axis. —— (5)

From eqs (4) and (5), we say that given curve is symmetric about polar axis.

SYMMETRIC ABOUT NORMAL AXIS

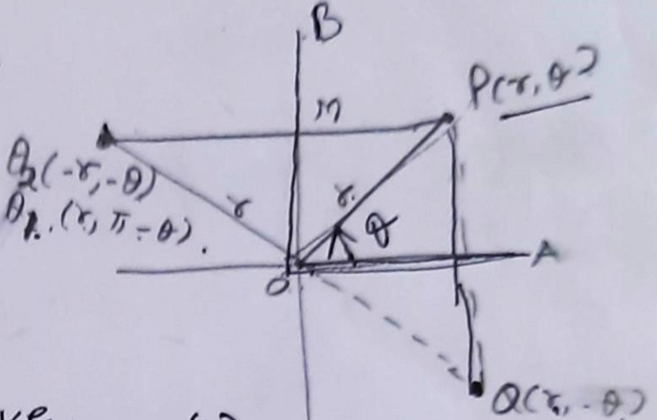
Theorem 2. Prove that a curve given by polar equation is symmetric with respect to normal axis, if one of the following condition hold.

(i) The equation remains unchanged on replacing θ by $\pi - \theta$

(ii) The equation remains unchanged on replacing r by $-r$ and θ by $-\theta$

OR

State when a polar curve is symmetric with respect to normal axis? Also prove it.



Proof: Let $f(r, \theta) = 0$ be a polar curve

(i) If equation remains unchanged on replacing θ by $\pi - \theta$, then $f(r, \pi - \theta) = 0$.

$\Rightarrow$ If $P(r, \theta)$ lies on curve then $Q_1(r, \pi - \theta)$ also lies on the curve. —— (1)

Let PQ_1 cuts normal axis at M.

ΔPOQ_1 is isosceles triangle with

$OP = OQ_1$ and OM bisect $\angle POQ_1$

$\Rightarrow OM$ is $\perp^r$ bisector of PQ_1.

point P and Q_1 are symmetric about normal axis. —(2)

From (1) and (2) we say that given curve is symmetric with respect to normal axis.

(ii) If the eq remains unchanged by replacing r by $-r$ and θ by $-\theta$, then we have

$$f(-r, -\theta) = 0$$

$\Rightarrow$ If $P(r, \theta)$ lies on the curve then $Q_2(-r, -\theta)$ will also lies on the curve. —— (3)

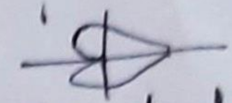
From fig we see that Q_1 and Q_2 are the same point. —— (4)

$\therefore$ from case (i) P and Q_2 are symmetric point about normal axis.

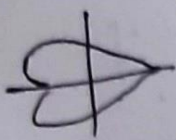
From (3) and (4) we say that curve is symmetric with respect to normal axis.

LIMACON

Curve given by equation $r = a \pm b\cos\theta$ or
$r = a \pm b\sin\theta$ $(a>0, b>0)$
are called Limacon.



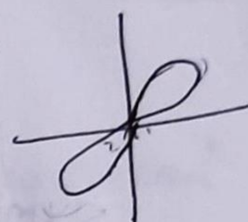
NOTE:



→ If $a=b$, then limacon is cardiod
→ If $a>b$, " " surround the pole.
→ If $a<b$; " " has inner loop.

LEMNISCATE

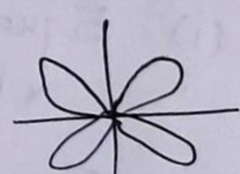
Curve given by $r^2 = \pm a^2\cos 2\theta$ or
$r^2 = \pm a^2 \sin 2\theta$
is called Lemniscate.



ROSE CURVE

Curve given by $r = a\cos n\theta$ or
$r = a\sin n\theta$ $(a>0, n\in N)$
is called rose curve.

If n is odd, Graph has n loops.
n is even, Graph has $2n$ loops.



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Exp. 01 Sketch the following curve
$r = 3(1+\cos\theta)$

Solution: Given that $r = 3(1+\cos\theta)$,
$= 3 + 3\cos\theta$

Comparing with $r = a + b\cos\theta$
we have $a = 3,\ b = 3$
$\therefore a = b$
Hence the given curve is cardiod.

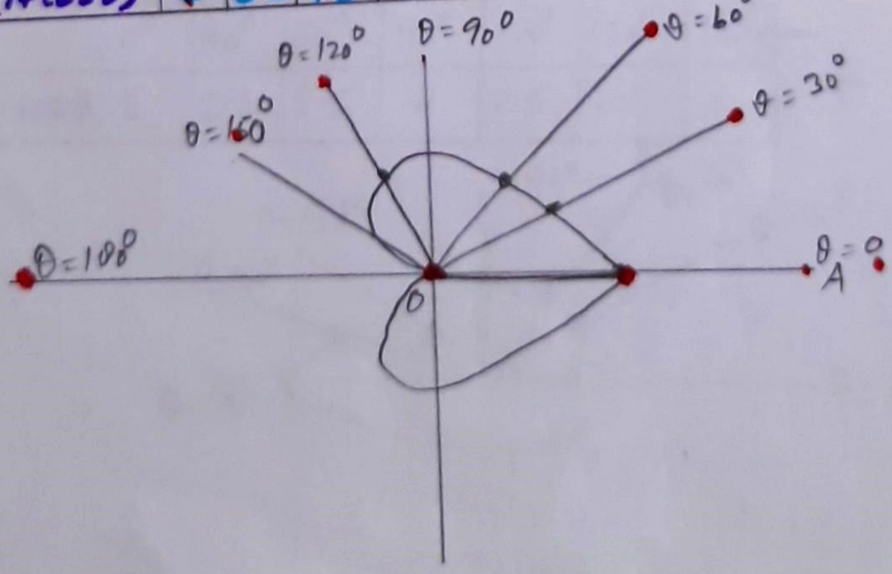
1. SYMMETRY Replace θ by $-\theta$
$r = 3(1+\cos(-\theta))$
$r = 3(1+\cos\theta)$
Equation is unchanged. ~~Symmetry~~ Curve is symmetric about polar axis.

2. Closeness Replace θ by $2\pi+\theta$
$r = 3(1+\cos(2\pi+\theta))$
$= 3(1+\cos\theta)$
Eq. is unchanged. Hence curve is closed.

3. Extent $-1 \le \cos\theta \le 1$
Adding 1, we get $0 \le 1+\cos\theta \le 2$
Multiplying by 3, we get $0 \le 3(1+\cos\theta) \le 6$
$0 \le r \le 6$

4. Table of some points Curve is symmetric about polar axis so we can take values of θ from 0 to π.

θ	0	30°	60°	90°	120°	150°	180°
$r = 3(1+\cos\theta)$	6	5.6	4.5	3	1.5	0.4	0



Exp. 02. Sketch the following curve

$$r = 2 - \cos\theta.$$

Solution Given that $r = 2 - \cos\theta$

Comparing with $r = a - b\cos\theta$

we get $a = 2$, $b = 1$

$\therefore a > b$

Hence the given curve is a limacon which surrounds pole.

1 SYMMETRY Replace θ by $-\theta$

$$r = 2 - \cos(-\theta)$$
$$= 2 - \cos\theta$$

Eq is not changed. Curve is symmetric about polar axis.

clearly curve is neither symmetric about normal axis nor about pole.

2. Closeness Replace θ by $2\pi + \theta$

$$r = 2 - \cos(2\pi + \theta)$$
$$= 2 - \cos\theta$$

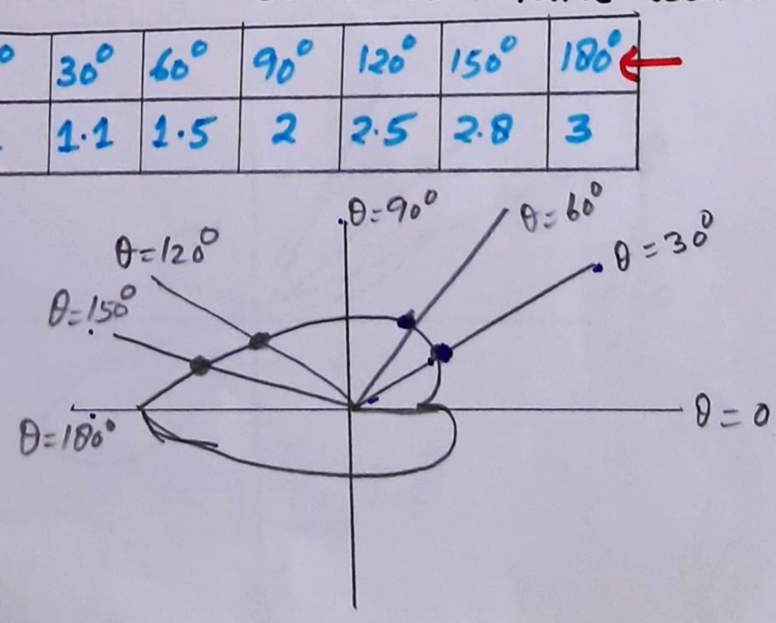
Eq. not changed. Hence curve is closed.

3. EXTENT $-1 \le \cos\theta \le 1$

$$1 \ge -\cos\theta \ge -1$$
$$2+1 \ge 2 - \cos\theta \ge 2 - 1$$
$$3 \ge 2 - \cos\theta \ge +1$$
$$1 \le 2 - \cos\theta \le 3$$
$$1 \le r \le 3$$

(4) Table of some points Curve is symmetric about polar axis so we can take value of θ between 0 to π.

θ	0°	30°	60°	90°	120°	150°	180°
$r = 2 - \cos\theta$	1	1.1	1.5	2	2.5	2.8	3



Exp 03 Sketch the following curve.

$$r^2 = 9\sin 2\theta$$

Solution Given that $r^2 = 9\sin 2\theta$, so given curve is lamniscate.

1. **SYMMETRY** Replace r by $-r$

$$(-r)^2 = 9\sin 2\theta$$
$$r^2 = 9\sin 2\theta$$

Eq not changed. Curve is symmetric about pole

2. **CLOSENESS** Replace θ by $2\pi+\theta$

$$r^2 = 9\sin 2(2\pi+\theta)$$
$$= 9\sin 2\theta$$

Eq not changed. Hence curve is closed.

3. **EXTENT** $-1 \le \sin 2\theta \le 1$

Multiplying by 9, we get

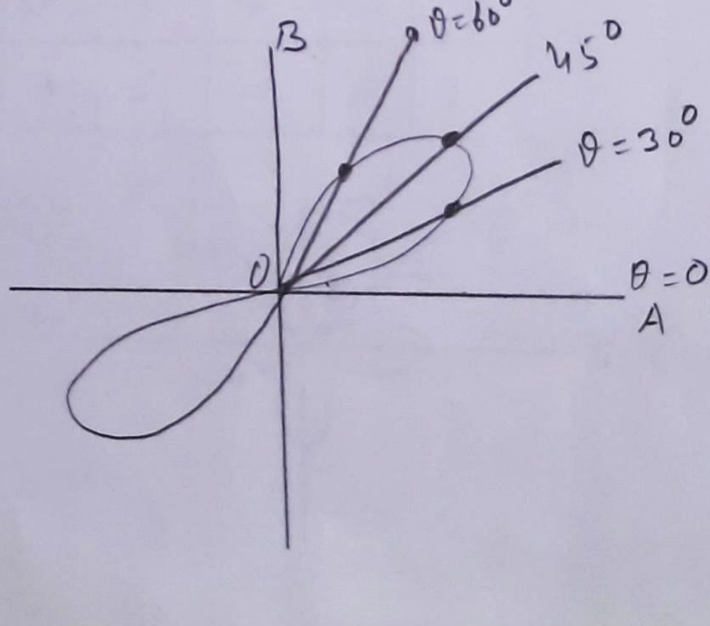
$$-9 \le 9\sin 2\theta \le 9$$
$$-9 \le r^2 \le 9$$
$$-3 \le r \le 3.$$

Also

$$r^2 \geq 0.$$
$$\Rightarrow 9\sin 2\theta \geq 0$$
$$\Rightarrow \sin 2\theta \geq 0.$$
$$\Rightarrow 0 \le 2\theta \le \pi \text{ or } 2\pi \le 2\theta \le 3\pi$$
$$\Rightarrow 0 \le \theta \le \pi/2 \text{ or } \pi \le \theta \le \frac{3\pi}{2}$$

4. **Table of some points** Given curve is symmetric about pole, so we can take θ from 0 to $\frac{\pi}{2}$.

θ	0°	30°	45°	60°	90°
$r^2 = 9\sin 2\theta$	0	7.8	9	7.7	0
$r = \pm 3\sqrt{\sin 2\theta}$	0	± 2.7	± 3	± 2.7	0



Exp 04 Sketch the following curve
$r = \cos 2\theta$

Solution Given curve is $r = \cos 2\theta$

Here Given curve is rose curve with **4 loops**.

1 SYMMETRY

1. Replace θ by $-\theta$, $r = \cos 2(-\theta) = \cos 2\theta$
Eq. is unchanged. $\therefore$ curve is symmetric about polar axis.

2. Replace θ by $\pi - \theta$, $r = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta)$
$= \cos 2\theta$. Hence eq. is unchanged
curve is symmetric about normal axis

3. Replace θ by $\pi + \theta$
$r = \cos 2(\pi + \theta) = \cos(2\pi + 2\theta) = \cos 2\theta$
Eq. is unchanged. Hence curve is symmetric about pole.

2. Closeness

Replace θ by $2\pi + \theta$

$r = \cos 2(2\pi + \theta) = \cos(4\pi + 2\theta)$
$= \cos 2\theta$

Eq. is unchanged. Hence curve is closed.

3. Extent

$-1 \le \cos 2\theta \le 1$

$-1 \le r \le 1$

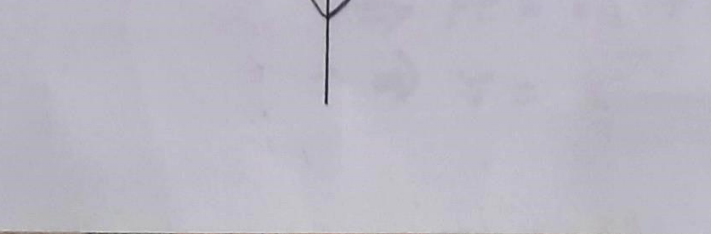
We Know that
$\cos 2\theta = 0$.
$2\theta = \pi/2$.
$\theta = \pi/4$.

$\cos 2\theta = 1$
$2\theta = 2\pi$
$\Rightarrow \theta = \pi$

4. Table

curve is symmetric about polar axis. so we can take θ between 0 to π and the difference between θ is $\frac{\pi}{4}$.

θ	0°	45°	90°	135°	180°
$r = \cos 2\theta$	1	0	-1	0	1



POLAR EQUATION OF CONIC

Thm. In usual notation prove that

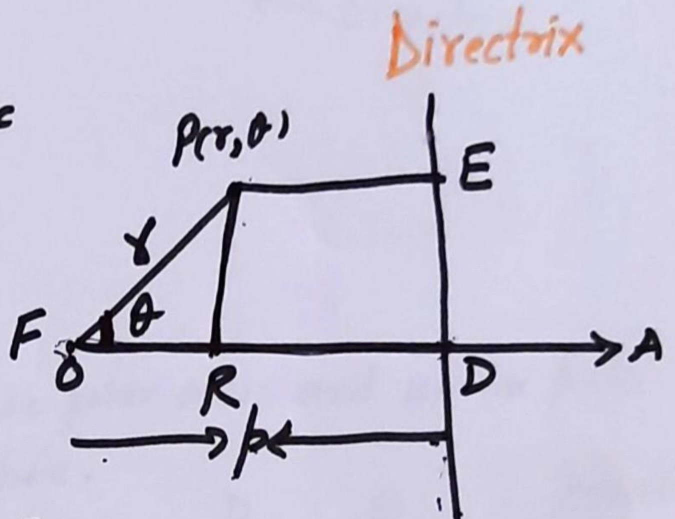
(i) $r = \dfrac{pe}{1 \pm e\cos\theta}$

(ii) $r = \dfrac{pe}{1 \pm e\sin\theta}$

Consider a conic whose one focus is at pole.
① obtain an equation of conic, where directrix is parallel to the polar axis
② obtain an equation of conic, where directrix is $\perp^r$ to the polar axis.

Proof Case-1 Directrix is $\perp^r$ to the polar axis and right to the pole at a distance p from the pole.

Let focus F be at pole O
Let $P(r, \theta)$ be any point on conic
Draw $PE \perp^r$ to Directrix
" $PR \perp^r$ to polar axis
" $OD \perp^r$ to Directrix



From fig $OP = r$, $\angle DOP = \theta$ and $OD = p$

From right angled triangle POR,

$$\cos\theta = \frac{OR}{OP} \Rightarrow OR = r\cos\theta$$

$$\sin\theta = \frac{PR}{OP} \Rightarrow PR = r\sin\theta$$

By def. of eccentricity

$$e = \frac{d(P, \text{focus } F)}{d(P, \text{Directrix})} = \frac{PF}{PE} = \frac{r}{RD} = \frac{r}{OD - OR}$$

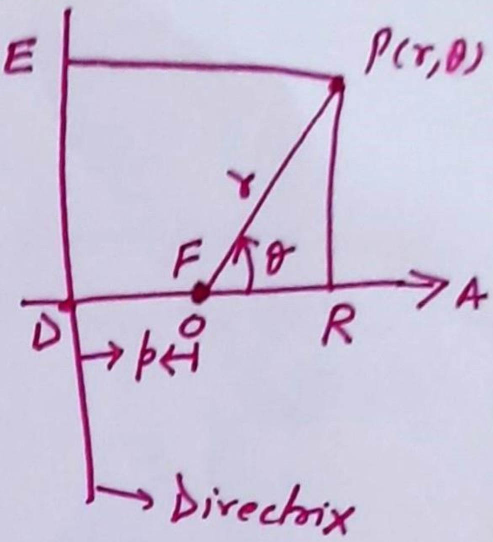
$$e = \frac{r}{p - r\cos\theta} \Rightarrow pe - re\cos\theta = r$$

$$\Rightarrow pe = r(1 + e\cos\theta)$$

$$\Rightarrow r = \frac{pe}{1 + e\cos\theta}.$$ **Proved**

Case. II If Directrix is $\perp^r$ to polar axis and left to the pole at a distance p from the pole, then

$$e = \frac{PF}{PE} = \frac{r}{RD}$$



$$= \frac{r}{OD + OR}$$

$$= \frac{r}{p + r\cos\theta}$$

$\Rightarrow ep + er\cos\theta = r$

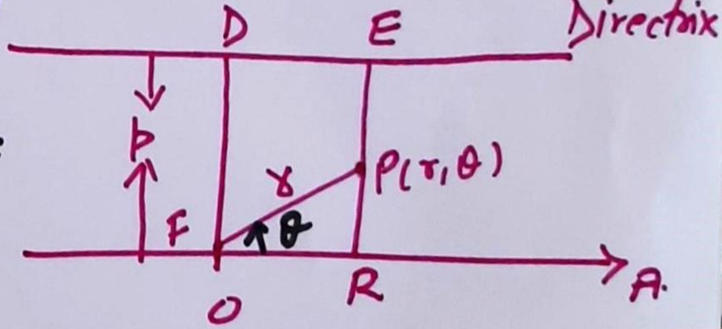
$\Rightarrow ep = r(1 - e\cos\theta)$

$\Rightarrow r = \dfrac{ep}{1 - e\cos\theta}$

Hence $r = \dfrac{ep}{1 \pm e\cos\theta}$

Proof-2.

Case-1 If directrix is parallel to polar axis and above pole at a distance p from the pole.



Let focus F be at pole.
Let $P(r,\theta)$ be any point on conic
Draw $PE \perp^r$ directrix
$OD \perp^r$ directrix
$PR \perp^r$ polar axis

Here $OP = r$, $\angle AOP = \theta$; $OD = p$

From $\Delta\, OPR$

$\cos\theta = \dfrac{OR}{r} \Rightarrow OR = r\cos\theta$

$\sin\theta = \dfrac{PR}{r} \Rightarrow PR = r\sin\theta$.

By def. of eccentricity

$$e = \frac{PF}{PE} = \frac{r}{ER - PR} = \frac{r}{p - r\sin\theta}$$

$$\Rightarrow ep - er\sin\theta = r$$

$$\Rightarrow ep = r(1+e\sin\theta)$$

$$\Rightarrow r = \frac{pe}{1+e\sin\theta}$$

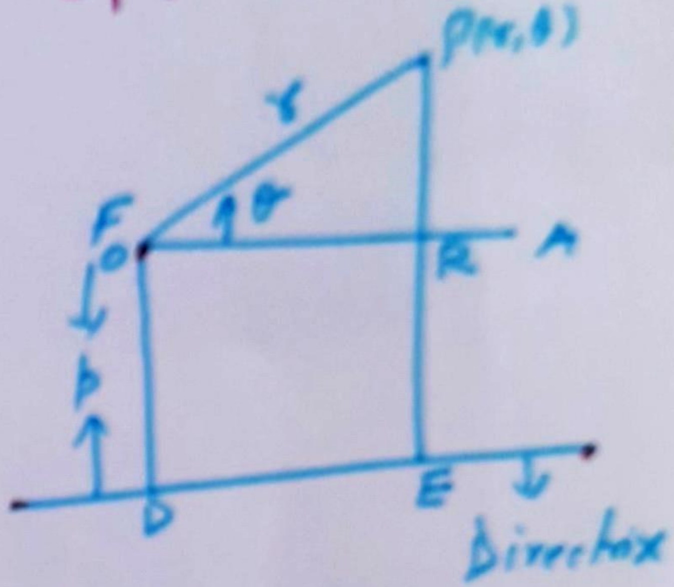
Case. II If directrix is parallel to polar axis and below the pole at a distance p from the pole.

We know that

$$e = \frac{PF}{PE}$$

$$= \frac{r}{PR+RE}$$

$$= \frac{r}{p+r\sin\theta}$$



$$\Rightarrow ep + er\sin\theta = r$$

$$\Rightarrow ep = r(1-e\sin\theta)$$

$$\therefore r = \frac{pe.}{1-e\sin\theta}$$

Hence $r = \dfrac{pe}{1 \pm e\sin\theta}$

———x———